whence by subtraction,

$$a'''-a''-(a'-a)-4B=6C,$$

and $$\delta a''-\delta a-2.\delta^2 a=6C;$$

or, since $$\delta a''-\delta a=\delta^2 a'+\delta^2 a=2.\delta^2 a+\delta^3 a,$$

we have $$C=\frac{\delta^3 a}{1.2.3}.$$

Finally, as the second equation gives

$$A=a''-a'-B-C,$$

we shall have $$A=\delta a'-\phi+\frac{1}{12}\cdot\delta^3 a,$$

and substituting these values of A, B, C, we get the formula (n).

It may be noticed that in using the formula (l), if we neglect altogether the last term which involves $\delta^3 a$, the second differences would still be partly corrected by the differences of the third order. In fact, the formula would then give

$$u_t=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{\alpha^2}\cdot\phi.$$

But,

$$\phi=\frac{1}{4}(\delta^2 a'+\delta^2 a)=\frac{1}{2}\left(\delta^2 a+\frac{1}{2}\delta^3 a\right)=\frac{1}{2}\left(\delta^2 a'-\frac{1}{2}\delta^3 a\right),$$

so that we shall have

$$u_t=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{1.2.\alpha^2}\left(\delta^2 a-\frac{1}{2}\delta^3 a\right).$$

It is from this last term that M. Mathieu has computed a table inserted in the *Connaissance des Temps*. Supposing $\alpha=12^h$, and that the motion of the star is uniform during those 12 hours, the two first terms $a'+\frac{t}{12^h}\cdot\delta a'$ are given; and we then apply the correction resulting from the last term by means of the table in question, which is computed for values of t differing by 10 minutes.

Generally speaking, in all these formulæ, we give to α such integral values as we may require, making it most frequently equal to 12^h, 6^h, 3^h, or 1^h.

On the Value of Apportionable (or Complete) Annuities. (*Continued.*)

By THOMAS B. SPRAGUE, M.A., *Actuary of the Equity and Law Life Assurance Society, and Vice-President of the Institute of Actuaries.*

HAVING thus found the required formula for the value of $\mathbf{a}^{(m)}_{k\overline{|\tau}}$, viz.:—

$$\mathbf{a}^{(m)}_{k\overline{|\tau}}=a_k+\frac{m+1}{2m}-\tau-\frac{\mu+\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)$$
$$-\frac{1}{12m^2}\frac{D'_k}{D_k}\tau(1-m\tau)(1-2m\tau)$$

we are now in a position to proceed with the transformation of the expression found in the last Number of this *Journal* (vol. xiii., p. 378), viz.:—

$$\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r-1}{mn}\right.}}-\frac{r}{n}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r}{mn}\right.}}.$$

In fact, making $\tau=\dfrac{r-1}{mn}$, and $\dfrac{r}{mn}$, successively, we get

$$\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r-1}{mn}\right.}}-\frac{r}{n}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r}{mn}\right.}}=\frac{r}{n}v^{\frac{1}{mn}}\left\{a_k+\frac{m+1}{2m}-\frac{r-1}{mn}-\frac{\mu+\delta}{12m^2}\left[m^2-1+6\frac{r-1}{n}-6\frac{(r-1)^2}{n^2}\right]-\frac{1}{12m^2}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\frac{r-1}{mn}\left(1-\frac{r-1}{n}\right)\left(1-2\frac{r-1}{n}\right)\right\}$$

$$-\frac{r}{n}\left\{a_k+\frac{m+1}{2m}-\frac{r}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+\frac{6r}{n}-\frac{6r^2}{n^2}\right)-\frac{1}{12m^2}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\cdot\frac{r}{mn}\left(1-\frac{r}{n}\right)\left(1-\frac{2r}{n}\right)\right\}.$$

This expression has now to be summed with respect to r, giving it the values, 1, 2, 3, ... n; and then n has to be supposed infinite. Rearranging the terms, we have

$$\Sigma\left\{\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r-1}{mn}\right.}}-\frac{r}{n}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r}{mn}\right.}}\right\}=-(1-v^{\frac{1}{mn}})\left\{\left(a_k+\frac{m+1}{2m}\right)\Sigma\frac{r}{n}-\frac{1}{m}\Sigma\frac{r^2}{n^2}-\frac{\mu+\delta}{12m^2}\left[(m^2-1)\Sigma\frac{r}{n}+6\Sigma\frac{r^2}{n^2}-6\Sigma\frac{r^3}{n^3}\right]-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left[\Sigma\frac{r^2}{n^2}-3\Sigma\frac{r^3}{n^3}+2\Sigma\frac{r^4}{n^4}\right]\right\}$$

$$+v^{\frac{1}{mn}}\left\{\frac{1}{m}\Sigma\frac{r}{n^2}+\frac{\mu+\delta}{2m^2}\left[\Sigma\frac{r}{n^2}-2\Sigma\frac{r^2}{n^3}+\Sigma\frac{r}{n^3}\right]+\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left[\Sigma\frac{r}{n^2}-6\Sigma\frac{r^2}{n^3}+3\Sigma\frac{r}{n^3}+6\Sigma\frac{r^3}{n^4}-6\Sigma\frac{r^2}{n^4}+2\Sigma\frac{r}{n^4}\right]\right\}\quad\ldots(31)$$

But since $\delta=-\log_e v$, or $v=e^{-\delta}$,

we have
$$v^{\frac{1}{mn}}=e^{-\frac{\delta}{mn}}=1-\frac{\delta}{mn}+\frac{\delta^2}{2m^2n^2}-\frac{\delta^3}{6m^3n^3}+\ldots.$$

and
$$1-v^{\frac{1}{mn}}=\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\frac{\delta^2}{6m^2n^2}-\ldots.\right)$$

Also
$$\Sigma\frac{r}{n}=\frac{1}{n}\frac{n(n+1)}{2}=\frac{n+1}{2}$$
$$\Sigma\frac{r^2}{n^2}=\frac{n(n+1)(2n+1)}{6n^2}=\frac{n}{3}+\tfrac{1}{2}+\frac{1}{6n}$$
$$\Sigma\frac{r^3}{n^3}=\frac{n^2(n+1)^2}{4n^3}=\frac{n}{4}+\tfrac{1}{2}+\frac{1}{4n}$$
$$\Sigma\frac{r^4}{n^4}=\frac{n}{5}+\tfrac{1}{2}+\frac{1}{3n}-\frac{1}{30n^2}.$$ (See vol. xi., p. 203.)

Then proceeding to the limit by making n infinite, and using the abbreviation "Lt" to denote "the limiting value, when n is supposed to be infinite, of," we have

$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r}{n}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{2}+\tfrac{1}{2}\right)$$
$$=\text{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\tfrac{1}{2}+\frac{1}{2n}\right)=\frac{\delta}{2m}$$
$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r^2}{n^2}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{3}+\tfrac{1}{2}+\frac{1}{6n}\right)$$
$$=\text{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\tfrac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)=\frac{\delta}{3m}$$
$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r^3}{n^3}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{4}+\tfrac{1}{2}+\frac{1}{4n}\right)=\frac{\delta}{4m}$$
$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r^4}{n^4}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{5}+\tfrac{1}{2}+\frac{1}{3n}-\frac{1}{30n^2}\right)=\frac{\delta}{5m}$$

Also, when n is infinite, $v^{\frac{1}{mn}}$ becomes equal to unity.

And in the same case

$$\text{Lt}\,\Sigma\frac{r}{n^2}=\text{Lt}\left(\tfrac{1}{2}+\frac{1}{2n}\right)=\tfrac{1}{2}$$
$$\text{Lt}\,\Sigma\frac{r^2}{n^3}=\text{Lt}\left(\tfrac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)=\tfrac{1}{3}$$
$$\text{Lt}\,\Sigma\frac{r^3}{n^4}=\text{Lt}\left(\tfrac{1}{4}+\frac{1}{2n}+\frac{1}{6n^2}-\frac{1}{30n^3}\right)=\tfrac{1}{4}.$$

We also see that

$$\text{Lt}\,\Sigma\frac{r}{n^3}=0,\quad \text{Lt}\,\Sigma\frac{r^2}{n^4}=0,\quad \text{Lt}\,\Sigma\frac{r}{n^4}=0.$$

Substituting these limiting values, we get the limit of the value of

$$\Sigma\left\{\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r-1}{mn}\right.}}-\frac{r}{n}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r}{n}\right.}}\right\}$$

when n is made infinite

$$=-\left\{a_k+\frac{m+1}{2m}\right\}\frac{\delta}{2m}+\frac{1}{m}\cdot\frac{\delta}{3m}+\frac{\mu+\delta}{12m^2}\left\{(m^2-1)\frac{\delta}{2m}+6\frac{\delta}{3m}-6\frac{\delta}{4m}\right\}$$

$$-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left\{\frac{\delta}{3m}-3\frac{\delta}{4m}+2\frac{\delta}{5m}\right\}$$

$$+\frac{1}{m}\cdot\tfrac{1}{2}+\frac{\mu+\delta}{2m^2}\{\tfrac{1}{2}-\tfrac{2}{3}\}+\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\{\tfrac{1}{2}-\tfrac{6}{3}+\tfrac{6}{4}\}$$

$$=\frac{1-\delta a_k}{2m}-\frac{\mu}{12m^2}-\frac{\delta}{4m}+\frac{(\mu+\delta)\delta}{24m}-\frac{\delta}{720m^4}\cdot\frac{\mathrm{D}''_k}{\mathrm{D}_k}\quad\ldots\ldots\quad(32)$$

and this ought therefore to be the value of the correction when the annuity is payable to the day of death.

Testing this result, however, by applying it to the case of a perpetual annuity certain, and making accordingly $\mu=0$, $\frac{\mathrm{D}''_k}{\mathrm{D}_k}=\delta^2$, $a_k=\frac{1}{i}$, it becomes

$$\frac{1}{2m}\left\{1-\frac{\delta}{i}-\frac{\delta}{2}+\frac{\delta^2}{12}-\frac{\delta^3}{360m^3}\right\}=\frac{1}{2m}\left\{\frac{\delta^4}{720}-\frac{\delta^3}{360m^3}\right\}\text{ nearly.}$$

The value in this case ought to vanish, and we thus see that the last term in (32) is not correct. The reason of this is, that in proceeding to the limit of (31), our process takes account of the small quantities of the third order (*i.e.* terms involving δ^3, $\frac{\delta\mathrm{D}''_k}{\mathrm{D}_k}$, &c.) in the first term, but not of the similar quantities in the second term. Instead of (32) we can only make use of the formula obtained by omitting the inaccurate last term; and indeed the above test shows that the correct formula will contain no term of the third order, so that the formula for the value of the correction will be,—accurately as far as terms of the third order inclusive

$$\frac{1-\delta a_k}{2m}-\frac{\mu}{12m^2}-\frac{\delta}{4m}+\frac{(\mu+\delta)\delta}{24m},$$

which is the same as we have already found (22). (Vol. xiii., p. 377.)

As a still further test of the accuracy of our results, let us now suppose that money bears no interest or $\delta=0$; then for a_k we must

substitute e_k, the curtate expectation of life, and the formula (23) gives as the value of the *complete expectation,* $e_k+\frac{1}{2}-\frac{\mu}{12}$, which is independent of m, as of course it ought to be, and agrees with the value found by Mr. Woolhouse (vol. xi. p. 328).

Since writing the preceding, I have noticed that Griffith Davies uses the phrase "complete annuity" to denote the annuity payable up to the day of death. This is analagous to the common phrase "complete expectation of life"; and seems preferable to the term I have hitherto employed in this paper.

The next problem which suggests itself is to find the value of an apportionable (or a complete) annuity payable m times a year, when the first payment, instead of being due at the time $\frac{1}{m}$, is due at the time τ, where τ is less than $\frac{1}{m}$.

Let each m-part be subdivided into n equal portions, and suppose $\tau=\frac{\sigma}{mn}$, where σ is supposed to be an integer. Whatever the value of τ, this can be done to any degree of accuracy required, by taking n and σ sufficiently large.

Suppose further that if death occur in the subdivisions numbered respectively

$$1,\quad 2,\quad 3,\ldots\ \sigma,\ \sigma+1,\ \sigma+2,\ldots\ n,\quad 1,\quad 2,\ldots\ \sigma,\ \sigma+1,\ldots$$

there is payable a sum

$$\frac{n-\sigma+1}{mn},\ \frac{n-\sigma+2}{mn},\ldots\frac{n}{mn},\ \frac{1}{mn},\ \frac{2}{mn},\ldots\frac{n-\sigma}{mn},\ \frac{n-\sigma+1}{mn},\ldots\ldots\frac{n}{mn},\ \frac{1}{mn},\ldots$$

then by supposing n infinite, we pass to the conditions of the problem proposed.

If now $\frac{\sigma}{mn}$ be added to each of the above payments, they become

$$\frac{n+1}{mn},\quad \frac{n+2}{mn},\ldots\frac{n+\sigma}{mn},\ \frac{\sigma+1}{mn},\ \frac{\sigma+2}{mn},\ldots\frac{n}{mn},\ \frac{n+1}{mn},\ldots\ldots\frac{n+\sigma}{mn},\ \frac{\sigma+1}{mn},\ldots$$

$$\text{or}\quad \left.\begin{matrix}\frac{1}{mn}\\ +\frac{1}{m}\end{matrix}\right\},\ \left.\begin{matrix}\frac{2}{mn}\\ +\frac{1}{m}\end{matrix}\right\},\ldots\left.\begin{matrix}\frac{\sigma}{mn}\\ +\frac{1}{m}\end{matrix}\right\},\ \frac{\sigma+1}{mn},\ \frac{\sigma+2}{mn},\ldots\frac{n}{mn},\ \left.\begin{matrix}\frac{1}{mn}\\ +\frac{1}{m}\end{matrix}\right\},\ldots\ldots\left.\begin{matrix}\frac{\sigma}{mn}\\ +\frac{1}{m}\end{matrix}\right\},\ \frac{\sigma+1}{mn},\ldots$$

Thus we see that the value of the correction is equal to the value of the correction already found (vol. xiii., p. 377)

+ the value of $\frac{1}{m}$ payable if death occur in one of the first σ subdivisions of any m-part

— the value of $\frac{\sigma}{mn}\left(=\tau\right)$ payable whenever death may occur.

$=\mathrm{P}+\mathrm{Q}-\mathrm{R}$, suppose.

Now to find the value of Q, we see that it is equal to

$$\begin{aligned}\frac{1}{m}\Big\{&v^{\frac{1}{mn}}(1-p_{k,\frac{1}{mn}})+v^{\frac{2}{mn}}(p_{k,\frac{1}{mn}}-p_{k,\frac{2}{mn}})+\dots\dots\dots\dots +v^{\frac{\sigma}{mn}}(p_{k\frac{\sigma-1}{mn}}-p_{k,\frac{\sigma}{mn}})\\&+v^{\frac{1}{m}+\frac{1}{mn}}(p_{k,\frac{1}{m}}-p_{k,\frac{1}{m}+\frac{1}{mn}})+v^{\frac{1}{m}+\frac{2}{mn}}(p_{k,\frac{1}{m}+\frac{1}{mn}}-p_{k,\frac{1}{m}+\frac{2}{mn}})+\dots+v^{\frac{1}{m}+\frac{\sigma}{mn}}(p_{k,\frac{1}{m}+\frac{\sigma-1}{mn}}-p_{k,\frac{1}{m}+\frac{\sigma}{mn}})\\&+v^{\frac{2}{m}+\frac{1}{mn}}(p_{k,\frac{2}{m}}-p_{k,\frac{2}{m}+\frac{1}{mn}})+\dots\dots\\&\vdots\\&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\Big\}\end{aligned}$$

$$=v^{\frac{1}{mn}}\left\{\mathbf{a}_k^{(m)}+\mathbf{a}^{(m)}_{k|\overline{\frac{1}{mn}}}+\mathbf{a}^{(m)}_{k|\overline{\frac{2}{mn}}}+\dots\dots+\mathbf{a}^{(m)}_{k|\overline{\frac{\sigma-1}{mn}}}\right\}-\left\{\mathbf{a}^{(m)}_{k|\overline{\frac{1}{mn}}}+\mathbf{a}^{(m)}_{k|\overline{\frac{2}{mn}}}+\dots\dots+\mathbf{a}^{(m)}_{k|\overline{\frac{\sigma}{mn}}}\right\}$$

$$=v^{\frac{1}{mn}}\left\{\mathbf{a}_k^{(m)}-\mathbf{a}^{(m)}_{k|\overline{\frac{\sigma}{mn}}}\right\}-(1-v^{\frac{1}{mn}})\left\{\mathbf{a}^{(m)}_{k|\overline{\frac{1}{mn}}}+\mathbf{a}^{(m)}_{k|\overline{\frac{2}{mn}}}+\dots\dots+\mathbf{a}^{(m)}_{k|\overline{\frac{\sigma}{mn}}}\right\}.$$

$$\begin{aligned}=v^{\frac{1}{mn}}\Big\{&a_k+\frac{m+1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)-a_k-\frac{m+1}{2m}+\frac{\sigma}{mn}+\frac{\mu+\delta}{12m^2}\left(m^2-1+6\frac{\sigma}{n}-6\frac{\sigma^2}{n^2}\right)\\&+\frac{1}{12m^2}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\frac{\sigma}{mn}\left(1-\frac{\sigma}{n}\right)\left(1-2\frac{\sigma}{n}\right)\Big\}\\-(1-v^{\frac{1}{mn}})\Big\{&a_k+\frac{m+1}{2m}-\frac{1}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+6\frac{1}{n}-6\frac{1}{n^2}\right)-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left(\frac{1}{n}-3\cdot\frac{1}{n^2}+2\cdot\frac{1}{n^3}\right)\\&+a_k+\frac{m+1}{2m}-\frac{2}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+6\cdot\frac{2}{n}-6\frac{2^2}{n^2}\right)-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left(\frac{2}{n}-3\frac{2^2}{n^2}+2\cdot\frac{2^3}{n^3}\right)\\&+\dots\dots\\&\vdots\\&+a_k+\frac{m+1}{2m}-\frac{\sigma}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+6\frac{\sigma}{n}-6\frac{\sigma^2}{n^2}\right)-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left(\frac{\sigma}{n}-3\frac{\sigma^2}{n^2}+2\frac{\sigma^3}{n^3}\right)\Big\}\end{aligned}$$

$$\left.\begin{aligned}=v^{\frac{1}{mn}}\Big\{&\frac{\sigma}{mn}+\frac{\mu+\delta}{2m^2}\left(\frac{\sigma}{n}-\frac{\sigma^2}{n^2}\right)+\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\frac{\sigma}{n}\left(1-\frac{\sigma}{n}\right)\left(1-\frac{2\sigma}{n}\right)\Big\}\\-(1-v^{\frac{1}{mn}})\Big\{&\sigma a_k+\sigma\frac{m+1}{2m}-\frac{\sigma(\sigma+1)}{2mn}-\frac{\mu+\delta}{12m^2}\left[\sigma(m^2-1)+\frac{3\sigma(\sigma+1)}{n}-\frac{\sigma(\sigma+1)(2\sigma+1)}{n^2}\right]\\&-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left[\frac{\sigma(\sigma+1)}{2n}-\frac{\sigma(\sigma+1)(2\sigma+1)}{2n^2}+\frac{\sigma^2(\sigma+1)^2}{2n^3}\right]\Big\}\end{aligned}\right\}(33)$$

Now proceed to the limit (as above, p. 38), supposing n to become infinite, $\frac{\sigma}{n}$ being equal to $m\tau$.

Then
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\,\sigma=\mathrm{Lt}\frac{\sigma}{n}\cdot\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)=\tau\delta$$
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\frac{\sigma(\sigma+1)}{n}=\mathrm{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)\frac{\sigma}{n}\left(\frac{\sigma}{n}+\frac{1}{n}\right)=\frac{\delta}{m}m^2\tau^2=m\delta\tau^2$$
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\frac{\sigma(\sigma+1)(2\sigma+1)}{n^2}=\mathrm{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)\frac{\sigma}{n}\left(\frac{\sigma}{n}+\frac{1}{n}\right)\left(2\frac{\sigma}{n}+\frac{1}{n}\right)=2\frac{\delta}{m}m^3\tau^3=2m^2\delta\tau^3$$
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\frac{\sigma^2(\sigma+1)^2}{n^3}=\mathrm{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)\frac{\sigma^2}{n^2}\left(\frac{\sigma}{n}+\frac{1}{n}\right)^2=\frac{\delta}{m}m^4\tau^4=m^3\delta\tau^4.$$

Hence the above expression (33) becomes
$$\tau+\frac{\mu+\delta}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{1}{12}\frac{D''_k}{D_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$
$$-\left\{\tau\delta\left(a_k+\frac{m+1}{2m}\right)-\frac{\delta\tau^2}{2}-\frac{\mu+\delta}{12m^2}[(m^2-1)\tau\delta+3m\delta\tau^2-2m^2\delta\tau^3]-\frac{1}{12m^3}\frac{D''_k}{D_k}\left[\frac{m}{2}\delta\tau^2-m^2\delta\tau^3+\frac{1}{2}m^3\delta\tau^4\right]\right\}$$
$$=\tau+\frac{\mu+\delta}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{1}{12}\frac{D''_k}{D_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)-\tau\delta a_k-\tau\delta\frac{m+1}{2m}+\frac{\delta\tau^2}{2}+\frac{\delta(\mu+\delta)\tau}{12}\left(1-\frac{1}{m^2}+3\frac{\tau}{m}-2\tau^2\right)$$
$$=\tau(1-\delta a_k)+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{\tau\delta}{2}\left\{\frac{1}{m}-\tau-\frac{m+1}{m}+\tau\right\}+\frac{1}{12}\delta(\mu+\delta)\tau\left(1-\frac{1}{m^2}+3\frac{\tau}{m}-2\tau^2\right)+\frac{1}{12}\frac{D''_k}{D_k}\tau\left(\frac{1}{m}-\tau\right)\left\{\frac{1}{m}-2\tau+\frac{\delta\tau}{2}\left(\frac{1}{m}-\tau\right)\right\}$$
or,
$$Q=\tau(1-\delta a_k)+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{\tau\delta}{2}+\frac{1}{12}\delta(\mu+\delta)\tau\left(1-\frac{1}{m^2}+3\frac{\tau}{m}-2\tau^2\right)+\frac{1}{12}\frac{D''_k}{D_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$
neglecting the term of the third order.

This, it will be remembered, is the value of $\frac{1}{m}$ payable at the instant of death, if it should occur during the time τ which commences any m-part of a year.

To test this, we see that making $\tau=0$, it vanishes, as it should.

We will further test it by applying it to an annuity certain. Making then $\mu=0$, $\frac{D''_k}{D_k}=\delta^2$, $a_k=\frac{1}{i}$, it becomes

$$\tau\left(1-\frac{\delta}{i}\right)-\frac{\tau\delta}{2}+\frac{\tau\delta^2}{12}=\tau\left(\frac{\delta}{2}-\frac{\delta^2}{12}\right)-\frac{\tau\delta}{2}+\frac{\tau\delta^2}{12}=0.$$

This also is correct; since in this case, there can be no death and therefore no payment.

If in the above value of Q, we suppose $\tau=\frac{1}{m}$, it is clear that we shall obtain the value of $\frac{1}{m}$ payable at the instant of death *whenever it may occur*. Making this substitution, Q becomes $\frac{1-\delta a_k}{m}-\frac{\delta}{2m}+\frac{\delta(\mu+\delta)}{12m}$; whence the value of £1 payable at the instant of death is

$$1-\delta a_k-\frac{\delta}{2}+\frac{\delta(\mu+\delta)}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

We have thus proved that

$$Q=\tau(1-\delta a_k)-\frac{\tau\delta}{2}+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{1}{12}(\mu+\delta)\delta\tau-\frac{\tau}{12}\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)\left\{\delta(\mu+\delta)-\frac{D''_k}{D_k}\right\}.$$

Also,

$$P=\frac{1-\delta a_k}{2m}-\frac{\delta}{4m}-\frac{\mu}{12m^2}+\frac{(\mu+\delta)\delta}{24m},$$

and from (34)

$$-R=-\tau(1-\delta a_k)+\frac{\tau\delta}{2}-\frac{1}{12}(\mu+\delta)\delta\tau.$$

$$\therefore\ P+Q-R=\frac{1-\delta a_k}{2m}-\frac{\delta}{4m}-\frac{\mu}{12m^2}+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{(\mu+\delta)\delta}{24m}-\frac{\tau}{12}\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)\left\{\delta(\mu+\delta)-\frac{D''_k}{D_k}\right\}.$$

This therefore is the value of the correction. Add now the value of the annuity, $\mathbf{a}^{(m)}_{k\overline{\tau}|}$,

or
$$a_k+\frac{m+1}{2m}-\tau-\frac{\mu+\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{1}{12}\frac{D''_k}{D_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

and we get the value of the complete annuity payable m times a year, the first payment being due at the time τ,

$$a_k\left(1-\frac{\delta}{2m}\right)+\frac{1}{m}+\frac{1}{2}-\frac{\delta}{4m}-\tau-\frac{\mu}{12m^2}+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{(\mu+\delta)\delta}{24m}-\frac{(\mu+\delta)}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{\tau}{12}\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)\delta(\mu+\delta)$$

$$=\left(a_k+\frac{1}{2}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{m}-\tau-\frac{\mu}{12m^2}+\frac{(\mu+\delta)\delta}{24m}-\frac{m^2-1}{12m^2}(\mu+\delta)+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{\mu+\delta}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{(\mu+\delta)\delta}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

$$=\left(a_k+\frac{1}{2}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{m}-\tau-\frac{\mu+\delta}{12}\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}-\frac{\delta}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{\mu+\delta}{12}\delta\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

$$=\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\left(\frac{1}{m}-\tau\right)\left\{1-\frac{\delta\tau}{2}-\frac{(\mu+\delta)\delta}{12}\tau\left(\frac{1}{m}-2\tau\right)\right\}\quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (35)$$

If now we test this result by making $\tau=\frac{1}{m}$, it becomes $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}$, which we see is correct by (23), vol. xiii., p. 377.

Next make $\tau=0$, and we get $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\frac{1}{m}$, which is also correct.

If, again, we make m infinite, we have $\tau=0$, since $\tau<\frac{1}{m}$, and the value becomes $a_k+\frac{1}{2}-\frac{\mu+\delta}{12}$.

Next, in the case of a perpetual annuity certain, (35) becomes

$$\left(\frac{1}{i}+\frac{1}{2}-\frac{\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\left(\frac{1}{m}-\tau\right)\left\{1-\frac{\delta\tau}{2}-\frac{\delta^2\tau}{12}\left(\frac{1}{m}-2\tau\right)\right\};\quad \text{since } \mu=0, \text{ and } \frac{D''_k}{D_k}=\delta^2.$$

This ought to agree with (27), vol. xiii., p. 380, when the same substitutions are made, *i. e.* with

$$\frac{1}{i}+\frac{m+1}{2m}-\tau-\frac{\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{\delta^2}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

$$=\frac{1}{i}+\frac{1}{2}+\frac{1}{2m}-\tau-\frac{\delta}{12}+\frac{\delta}{12m^2}-\frac{\delta\tau}{2}\left(\frac{1}{m}-\tau\right)-\frac{\delta^2}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

The difference between these expressions is

$$\left(\frac{1}{i}+\frac{1}{2}-\frac{\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{2m}-\frac{1}{i}-\frac{1}{2}+\frac{\delta}{12}$$

which is easily seen to vanish by means of the value of $\frac{1}{i}$ given by (15), vol. xiii., p. 371.

The value of the perpetual annuity certain in this case is

$$\frac{1}{m}\left\{v^{\tau}+v^{\tau+\frac{1}{m}}+v^{\tau+\frac{2}{m}}+\ldots\ldots\right\}=\frac{1}{m}\cdot\frac{v^{\tau}}{1-v^{\frac{1}{m}}}$$

$$=\frac{1}{\delta}+\frac{1}{2m}-\tau+\delta\left(\frac{1}{12m^2}-\frac{\tau}{2m}+\frac{\tau^2}{2}\right)-\frac{\delta^2}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)+\frac{\delta^3}{24}\left\{\tau^2\left(\frac{1}{m}-\tau\right)^2-\frac{1}{30m^4}\right\}-\ldots\ldots$$

Lastly, make $\tau=\frac{1}{2m}$; then (35) becomes

$$\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\frac{1}{2m}\left(1-\frac{\delta}{4m}\right)=\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{2m}-\frac{\delta}{24m^2}$$

and here making $m=1, 2, 4$, successively, we get the correct formulæ for the values of complete annuities,

payable yearly $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2}\right)+\frac{1}{2}-\frac{\delta}{24}=\left(a_k-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2}\right)+1-\frac{7}{24}\delta$ . . . . (36)

" half-yearly $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{4}\right)+\frac{1}{4}-\frac{\delta}{96}=\left(a_k-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{4}\right)+\frac{3}{4}-\frac{13}{96}\delta$ . . . . (37)

" quarterly $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{8}\right)+\frac{1}{8}-\frac{\delta}{384}=\left(a_k-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{8}\right)+\frac{5}{8}-\frac{25}{384}\delta$ . . . (38)

These will be the proper formulæ for calculating accurately the liability at any time of an Insurance Company in respect of the annuities it has granted—these annuities being, in practice, always payable up to the day of death.